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PROGRESS REPORT.

The following gentlemen have been admitted as members of the Society:—

N. Sundaram Aiyer, Esq., M.A., Assistant Professor of Mathematics, St. Joseph's College, Trichinopoly;

V. Raghavachariar, Esq., M.A., Professor of Mathematics, St. Joseph's College, Trichinopoly;

P. Krishnamachary, Esq., M.A., L.T., Assistant Lecturer in Mathematics, Ceded Districts College, Anantapur.

The following note taken from "Science," Vol. LIV. No. 1402, p. 456, is inserted here to give our members an idea of the mathematical activities in America and of the interest taken by the general public of that great Country in Mathematical Research:—

"As Trustee of the Edward C. Hegeler Trust Fund, Mrs. Mary Hegeler Carus, of La Salle, Illinois, recently promised to make the Mathematical Association of America a yearly contribution of twelve hundred dollars for five years to be used for the publication of mathematical monographs under the auspices of this Association. As is well-known the publication of scientific literature has been much hampered in recent years by the greatly increased cost of publication. Hence this gift is especially timely and noteworthy.

The letter confirming this gift was addressed to Professor Slaught, of the University of Chicago, and includes the following significant statement:—

If at the end of five years this project shall have proved successful it is my intention to then give to the Association a permanent endowment fund, and I will so direct my legal representatives, which will yield at least twelve hundred dollars annually.

As the great success of the project seems practically assured in view of the wide and deep interest already manifested therein on the part of leading mathematicians, the Mathematical Association of America seems to have good reasons for expecting a substantial permanent endowment to aid it in the furtherance of its great cause of improving collegiate mathematics.

There are now three national mathemetical organizations in America. The oldest of these is the American Mathematical Society, which was organized in 1888 as the New York Mathematical Society, but was reorganized about six years later under its present name. This Society devotes most of its energies to mathematical research, and, to further this cause, Professor L. L. Conant, who died in 1916, bequeathed to it ten thousand dollars, subject to Mrs. Conant's life interest, the income of which is to be offered once in five years as a prize for original work in pure mathematics.

The Mathematical Association of America was organized in 1915 with a view towards supplementing the work of the American Mathematical Society along the line of collegiate teaching. It has always collaborated with the Society holding joint meetings with it and having a large common membership. The gift announced above will make it possible to collaborate still more effectively in promoting the interests of Advanced Mathematics in this country. The National Council of Teachers of Mathematics, organized in 1920, is mainly devoted to the interests of the teaching of Secondary Mathematics and hence represents more distinctly a separate field, but it too has already begun to co-operate with the Mathematical Association of America.

The latter organization took steps several years ago towards the publication of a modern mathematical dictionary and has a standing committee on this subject. It has, however, not yet been able to push this laudable enterprise on account of lack of funds. The difficulty of such a work is increased by the fact that at present there exists no good mathematical dictionary in any language, and hence most of the material for such a work has to be collected from original sources."

18, Pycroft's Road,
Triplicane, Madras.

P. V. SESHU AIYAR,
Honorary Joint Secretary.

A STATISTICAL STUDY OF SOME EXAMINATION MARKS.*

BY P. V. SESHU AYYAR AND S. R. RANGANATHAN.

INTRODUCTORY.

For some time past, it has seemed to the authors that much valuable information, regarding certain problems connected with public examinations conducted on a large scale, might possibly be obtained by statistical investigations of the results of these examinations and their formulation in the shape of ready-made rules for tackling such problems.

It is well-known that, with all the care that is bestowed upon it, the standard of the question paper is not the same from year to year, neither can the valuation of answers be regarded as standardised. It is therefore very desirable, that some method must be found to make due allowance for these unavoidable variations (in standards) so that candidates may not suffer and the value of the examination as a test of fitness may remain steady.

Further, from the pedagogical standpoint, it would be of interest to get some quantitative measure of the correlation of the candidates' attainments in the various subjects.

With a view to throw some light upon these and similar problems the authors have taken up the present statistical study of the marks obtained by candidates in successive years in a certain public examination in the Madras Presidency.

MATERIAL AND METHOD.

The material on which this study is based is taken from the register of the marks assigned in a certain public examination in English, Mathematics, Physics, Chemistry and History in six consecutive years. Since the number of candidates was large, the authors have considered only the marks of the first 500 candidates, which may well be taken to be a satisfactory random sample.

In the calculation of the constants, G. F. Hardy's summation method of calculating the moments has been used. Since the correlations considered are all practically linear, the correlation co-efficient has been found by the usual product-moment formula of Karl Pearson.

The intervals employed for grouping the marks cover a range of 5 marks, and the ends of intervals are so chosen that the minimum for a

* Read before the Indian Science Congress, 1922. The authors are indebted to Prof. E. B. Ross of the Madras Christian College for valuable criticisms and suggestions.

pass in each of the subjects falls in the centre of each interval. Hence any moderation, that might have been made by the examiners in the neighbourhood of the minimum, will not have any effect on the main results of the investigation.

The tables which form the basis of this study are 42 in number. For want of space, Tables Nos. 35 and 42 alone are given here as samples. They show the correlations of the attainments of the candidates of the 5th year in Physics and Chemistry and those of the 6th year in English and History.

SAMPLE TABLES :

5th Year. Correlation-Table No. 35.

Physics.

Chemistry.

Marks	3—7	8—12	13—17	18—22	23—27	28—32	33—37	38—42	43—47	48—52	53—57	58—62	63—67	68—72	Total.
0—2	1														1
3—7		1	1	1		1									4
8—12	1	1	3	4	1	2									12
13—17			1	5	2	4	3	1	1						17
18—22		1	1	7	7	1	9	3							29
23—27			3	8	7	13	9	4							44
28—32			3	4	10	14	12	13	4	1	1				62
33—37			2	6	6	15	14	15	12	3	1		2		76
38—42				2	9	12	11	11	9	5	3	1			63
43—47					2	4	13	19	11	8	1		1	1	60
48—52					2	5	11	12	8	7	8	1			54
53—57						1	7	7	9	4	4	2			34
58—62							5	7	5	2	4	2	1		26
63—67							2	1	2	3	1	1	1		11
68—72									2	1	1	1		1	6
73—77											1				1
Total.	2	3	14	37	46	72	96	93	63	34	25	8	5	2	500

6th Year. Correlation-Table No. 42.

History.

English.

Marks.	8—12	13—17	18—22	23—27	28—32	33—37	38—42	43—47	48—52	53—57	58—62	63—67	Total.
13—17			2	1									3
18—22		1	5	6	7	5	1	2					27
23—27	1	2	7	14	12	12	17	7	4				76
28—32		4	3	10	22	44	31	17	2	1	1		135
33—37			3	6	16	31	35	36	9				136
38—42			1	1	10	23	41	44	17	8	1		146
43—47		1			6	13	19	20	10	7	5	1	82
48—52						1	3	8	9	3	1		25
53—57						2		2	2	2	2		10
58—62								1	1		1		3
63—67												1	1
68—72									1				1
Total.	1	8	21	38	73	131	147	137	55	21	11	2	**645**

FREQUENCY CURVES.

The frequency curves obtained possess small skewness in almost all cases. The skewness does not exceed ·25 in any of the cases considered. Values less than ·1 are more frequent. The criterion k of Karl Pearson also has small numerical values. (It is frequently negative). The Pearsonian curves of Types I and IV are among those that give the best fit to the frequency curves. Type IV is of more frequent use. A few typical curves are shown in Figures I (*a*) and I (*b*).

RESULTS.

The main constants—the Median, the Mean, and the Standard Deviation (reduced to a maximum of 100)—for the various years and subjects are tabulated here. They are also represented graphically in Figs. II and III. As may be expected, the Standard Deviation is more stable than the Mean.

TABLE I.

	Year.	Median.	Mean.	Std. Devn.
English.	1	38·1	38·7	9·15
	2	39·7	40·4	9·33
	3	38·9	39·7	9·16
	4	42·3	43·2	10·15
	5	37·3	37·3	9·43
	6	35·8	36·3	8·31
Mathes.	1	34·6	35·3	13·67
	2	42·4	43·6	13·22
	3	42 2	43·8	12·89
	4	39·6	40·7	12·32
	5	45·3	46·3	13·65
	6	37·0	39·0	13·64
Physics.	1	40·1	40·6	14·43
	2	41·7	41·4	11·12
	3	44·8	44·9	11·65
	4	36·0	36·4	11·57
	5	36·1	36·5	11·06
	6	41·1	41·7	12·07
Chemistry.	1	...	35·8	14·34
	2	42·7	42·3	12·10
	3	40·5	41·6	11·83
	4	41·0	41·2	12·92
	5	37·9	38·4	13·69
	6	35·8	35·7	10·12
History.	1	37·1	37·5	7·95
	2	34·6	34·3	7·61
	3	40·3	40·9	9·93
	4	39·5	40·3	8·84
	5	37·0	36·4	8·29
	6	39·2	38·8	8·96

DISCUSSION OF THE RESULTS.

I. ADJUSTMENT OF THE MINIMUM.

It now remains to consider certain very important problems suggested by the numbers given in the foregoing Tables. Figs. II and III bring out prominently the fluctuations in the Mean and in the Standard Deviation of the marks. These fluctuations cannot be due to "probable error" alone. For, from Table V of Karl Pearson's "Tables for Biometricians and Statisticians," for the size of the population considered, we get

Fig. I (*a*)

FREQUENCY CURVES.

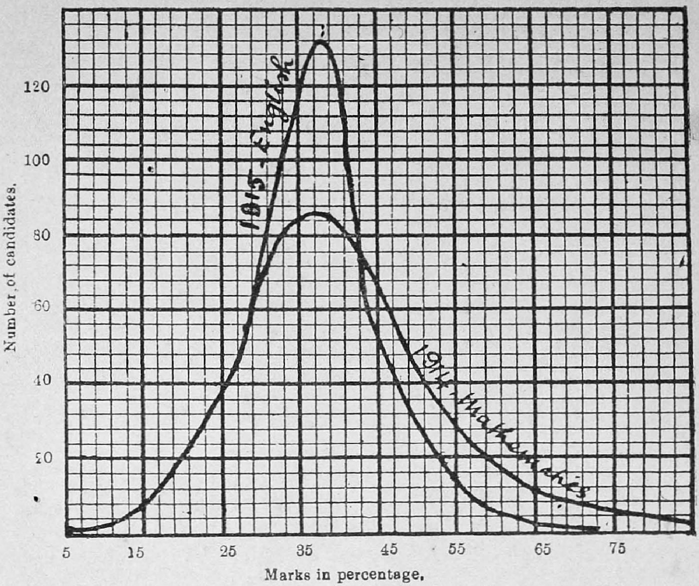
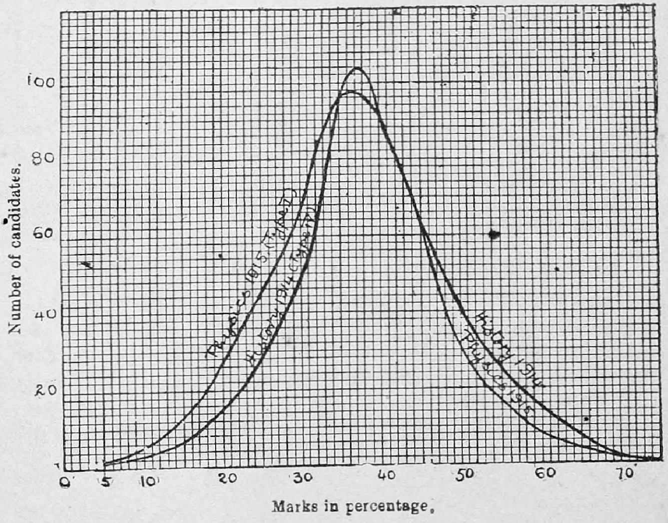


Fig. I (*b*)

FREQUENCY CURVES.



for the probable error of the Mean ·03016 times the Standard Deviation and for that of the Standard Deviation ·02133 times the Standard Deviation. Now, in the cases considered, the Standard Deviation does not exceed 14·5. Hence, we get for the greatest possible *probable error* ·42 in the case of the Mean and ·31 in the case of the Standard Deviation. But, the actual flctuations are considerably beyond these limits. Hence, we have to conclude that they may be considered as *real* fluctuations.

On the other hand, it can be assumed as a very convenient and highly reasonable hypothesis, that the average intelligence as well as the dispersion of the intelligence of the candidates as measured respectively by the Mean and the Standard Deviation of the marks in the several subjects, will be constant from year to year. For, if the observed fluctuations in the Mean and the Standard Deviation are due to variations in the average intelligence, they will be similar in the various subjects. But, the Figs. II and III indicate no such similarity. Hence, these fluctuations can be attributed mainly to fluctuations in the standard of the examination as determined by the difficulty of the question papers, the stringency of the marking, etc.

It may be urged that these fluctuations may also be due to sudden changes in the efficiency of school-training, due to change in the Regulations. Then, the preparation of candidates and the examination papers have to fit one another as a key fits a lock. Whether this fact can explain the fluctuations to a large extent, can be finally decided only by carrying on the investigation for a further period of, say, 5 or 6 years preceding and succeeding the year in which the changes in the regulations were introduced.

Now in order that the worth of the candidates may be judged, from year to year, by an equitable and uniform standard, it is highly desirable that suitable adjustments of the marks of the candidates should be made every year based upon the deviations of the Mean and Standard Deviation of the marks for the year from what may be deemed to be their normal values. For these normal values, we can take the averages of the Mean and the Standard Deviation for a number of consecutive years, as is done in meteorological statistics.

1. The Normal Mean and the Normal Standard Deviation, as determined by the averages for the six years considered are given here for reference.

TABLE II.

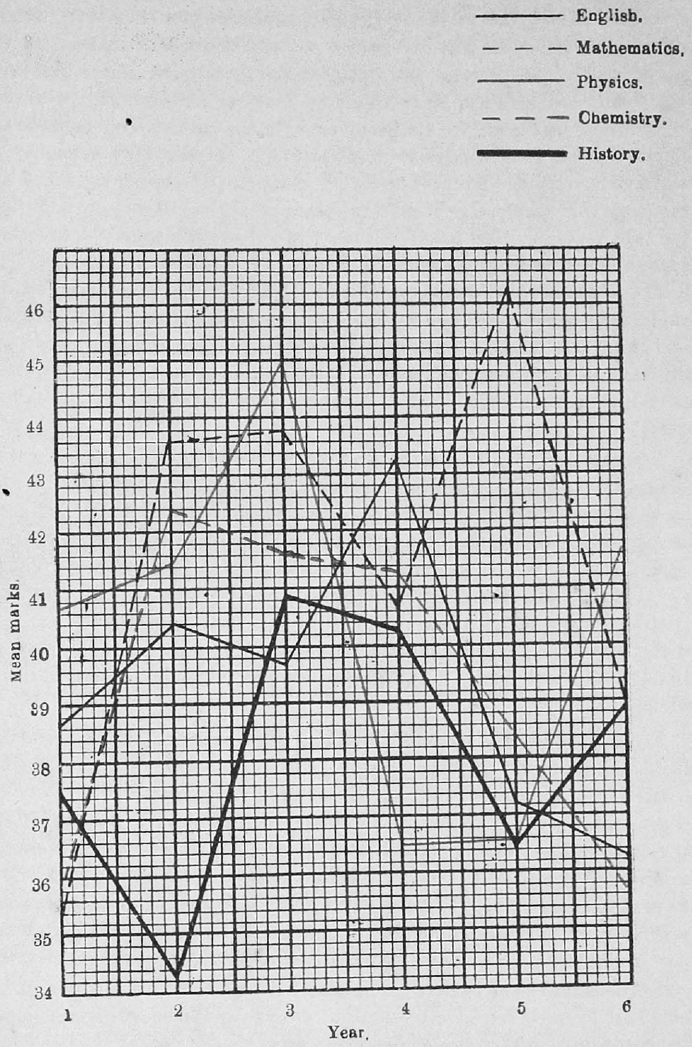
Year.	English.	Mathematics.	Physics.	Chemistry.	History
Normal Mean ...	39·3	41·5	40·3	39·1	38·0
Normal Std. Devn.	9·26	13·23	11·98	12·5	8·60

[NOTE.—If the marks of the candidates are entered in their certificates, a better plan will be to enter the adjusted marks instead of the actual marks. In practice, the actual marks are entered side by side with the Mean. Entering the Mean also is no doubt an improvement. But still, this is not sufficient for two reasons: (1) the adjustment due to the fluctuation in standard deviation is omitted; (2) there is a tendency in the layman to judge a candidate by the mark that is entered as earned by him, without comparing it with the Mean that is entered by its side. Perhaps a simple illustration will make this clear. Suppose in a certain year the Mean is 46 and a candidate earns 40. And in another year the Mean is 33 and a candidate earns 36. There is, we believe, a tendency in the public to be carried away by the fact that 40 is greater than 36 and conclude that the first candidate is better than the second, which is not true. At any rate, it can be said that it would be much safer to enter the adjusted mark instead of *the Mean*, and it would not involve a prohibitive amount of work.]

In the case of the examinations, where marks are not entered in the certificate it is not necessary in practice to make the adjustments in the marks of each candidate. Since the candidates are generally divided into three classes, namely, the first class containing those that get, say, 60 per cent. of the marks, the second class containing those that get an ordinary pass, and the third class containing the failures, and since the really serious question (except perhaps in the case of the top-men) is into what class a particular candidate falls and not so much what his rank is in that class, it would be sufficient if the minima for the first class and the second class are suitably adjusted.

The adjustment of the minimum for a pass is discussed here in detail. The minimum for a first class may be similarly adjusted. Let us take, as an illustration, the case of English. The minimum prescribed for a pass is 40 per cent. Now, it would be equitable to insist every year upon a candidate getting 40 per cent., only if the mean and the standard deviation of the marks continue to be the same from year to year. If, however, they deviate, in any particular year, from their normal values, as they do in practice, the equitable, minimum must be fixed a little higher or lower as the case may be. Thus, take the 6th year out of the years considered by us. The actual mean in that year is 36·3; but the normal mean is seen from Table II to be 39·3. We have assumed the average intelligence to be constant from year to year, and so this deviation of 3 from the normal may be taken as due to the overstrictness of the examination that year. Since the average candidate has been given 3 marks less in this year, we must reduce the minimum for a pass also proportionately. The minimum and the normal mean are practically the same, and as a result of this adjustment the

FIG. II. FLUCTUATIONS OF THE MEAN.



minimum should be brought down from 40 to 37. Further the normal dispersion of the marks corresponds to a standard deviation of 9·26 whereas the actual standard deviation in the year is 8·31. This means that there has been more than normal crowding about the mean, *i.e.* each candidate has been put a bit nearer to the mean, than he ought to be. Hence the minimum also should be brought nearer to the mean to the same extent. Hence the distance of the reduced minimum, *viz.* 37 from the actual mean must be decreased in the ratio of 8·31 to 9·26, which gives, as the final adjusted minimum, 36·93 per cent.

A similar adjustment in Chemistry in the 6th year would bring the minimum from 35 per cent. to 32·38 per cent. and in the case of mathematics in the 5th year the minimum would be raised from 35 to 39·60 per cent.

The working rule for the adjustment of the minimum may be enunciated as follows :—

Add the excess (or proportionate excess if the normal mean is not very near the minimum) of the actual mean over the normal mean to the prescribed minimum. Increase the distance of this result from the actual mean in the ratio of the actual standard deviation to the normal standard deviation. Instead of adding the excess of the actual mean to the minimum, we may also increase the minimum in the ratio of the actual mean to the normal mean. There will not be much appreciable difference between the results got by these two methods, in the cases that we are considering, where the normal mean is very near the minimum.

II. MARGIN OF PROBABLE ERROR.

Next, we have to consider what margin of probable error must be allowed in the neighbourhood of the minimum. For, justice requires that candidates whose marks are lower than the adjusted minimum by less than the probable error must be given the benefit of doubt.

Professor Edgeworth has discussed this problem rather fully* and has given the following results as his considered opinion based on several inquiries and experiments conducted with the aid of a number of examiners. In the absence of similar elaborate investigations pertaining to Indian conditions, we are justified in adopting his results.

The causes for the probable error in marking have been analysed and brought under five " heads" :—

(i) *minimum sensible* : " Error due to the difference of perception of excellence whose magnitude varies with the subject, being least in Mathematics and perhaps greatest in Composition." " Where subjects

* Vide : *Journal of the Royal Statistical Society, 1888.*

are for the most part of the simpler sort, it will perhaps be sufficient to allow for this source of uncertainty a probable error of 5 per cent. on the mark for each answer." We may take it that Mathematics is a subject "of the simpler sort." In the case of the other subjects, the error due to this cause is bound to be higher. 'Properly speaking we must determine it by actual experiment for each subject. It is observed that in a subject, in which we expect this error to be great, the standard deviation is smaller. We have taken 7 per cent. as the error in Physics and Chemistry and 10 per cent. as the error in English and History.

(ii) *personal equation* : "10 per cent. on the mark of each answer."

(iii) *difference in the scale adopted by the several assistant examiners* : "4·5 per cent. for each paper."

(iv) *fatigue of the examiner* : "1·5 per cent. for each paper."

(v) *speed of valuation* : "25 per cent. on the mark of each question."

To find the aggregate error in each subject due to all these causes, we have to take into consideration (i) the number of papers in the subject, and (ii) the most probable number of questions, in each paper, that will be attempted by the candidates who are in the border line between pass and failure. This has to be decided only by the experience of examiners. We have made the following assumptions for the examination, the results of which we are considering :—

English ...	I paper	5 out of 7 questions.
	II "	4 out of 5 "
	III "	3 out of 3 "
	IV "	3 out of 3 "
	Total ...	15

Mathematics (two papers)	...	5 out of 7 or 8 questions.
Physics "	...	5 out of 6 questions.
Chemistry "	...	4 out of 5 questions.
History "	...	5 out of 5 questions.

On this assumption, and neglecting the error due to speed, we get by the usual formula,* for the aggregate probable error due to all these causes, the following percentage of the marks earned by a border line candidate :—

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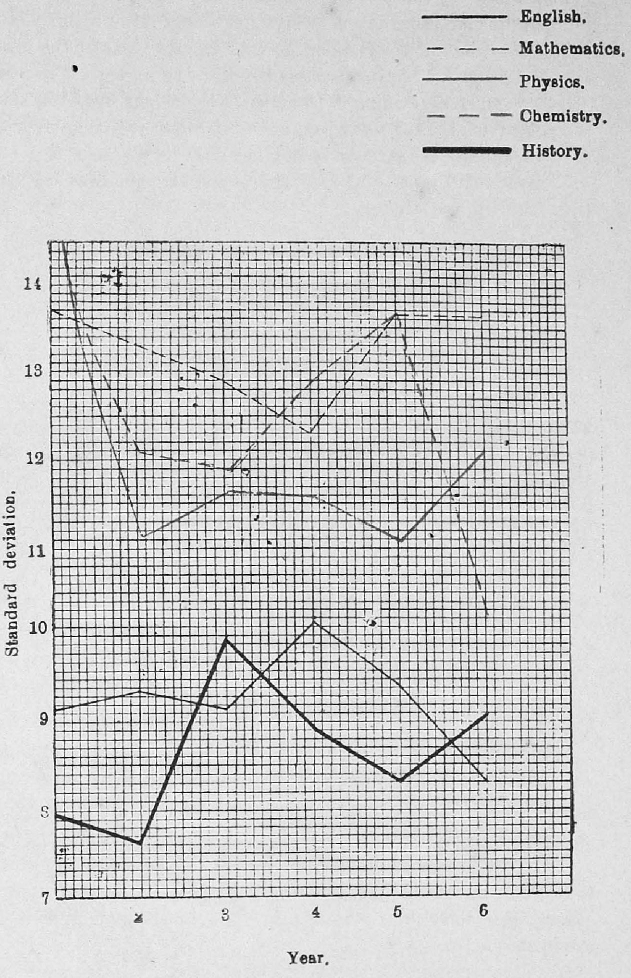
* (1) If A is the *p.e.* in valuing 1 question, the *p.e.* in valuing n questions is $\frac{A}{\sqrt{n}}$.

(2) If E_1 is the error due to one cause, E_2, the error due to another cause and so on, the aggregate error is $(E_1{}^2 + E_2{}^2 + \ldots\ldots)^{\frac{1}{2}}$.

Fig. III.

FLUCTUATIONS OF THE STANDARD DEVIATION.



English.	Mathematics.	Physics.	Chemistry.	History.
4·4	4·8	5·1	5·1	5·8

Going back to the illustration we have had already, *viz.* English and Chemistry in the 6th year, the adjusted minima in those subjects, less the probable error, come to 35 3 per cent. and 30 7 per cent. respectively. Consequently taking the maximum in these subjects to be 300 and 100 respectively, all those that get 106 in English and 31 in Chemistry must be passed. To bring out the real significance and importance of these two adjustments, it may be mentioned that in the year under consideration, out of 500 candidates, about 50 more would have passed in each of the subjects, English and Chemistry if the marks had been moderated as suggested. Taking both the subjects together, about 30 more would have passed. If the adjustment were made in the other subjects also, there would no doubt be some more addition to the number of passes.

In the above calculation of the probable error, it has been assumed that the valuation of the papers was done very leisurely without allowing any error due to speed to creep in. If, however, this assumption is not tenable, we must make allowance for speed also, that would still further increase the margin of probable error. The percentage in English, for example, works out to be 7·8 per cent. of the marks earned by the candidates.

Besides these errors due to valuation, there is still another error, *viz.* that due to the intrinsic variation in the performance of the candidate. The idea of this is as follows :—Two question papers are adjusted to be identical in standard so as to produce the same mean and standard deviation. A candidate tries both. He will not, even under an ideal system of marking, get the same mark in each. If he took a large number of such papers, there would be a standard deviation in the marks earned.

The margin of error that must be allowed for this source has not been investigated.

III. THE FIXING OF THE MINIMUM.

Then comes the problem of fixing the minimum for a pass. The fixing of the minimum appears to be very arbitrary at present. The arbitrariness is brought out by a comparison of the minima in the various subjects with the respective means. We have the following figures in percentages :—

	English.	Mathcs.	Physics.	Chemistry.	History.
Normal Mean	39·3	41·5	40·3	39·1	38
Minimum ...	40	35	35	35	35

It may be seen that for a pass less than average intelligence is demanded in science and history, whereas slightly more than average intelligence is

demanded in English. To appreciate the effect of this, we have to recollect that Table I shows that the median practically shadows the mean, *i e.* to say, roughly about half the candidates get less than the mean mark in almost all the subjects. So that fixing the minimum as 40 per cent. in English postulates the failure of slightly more than 50 per cent. of the candidates. Whereas the minimum of 35 per cent. in the other subjects implies the failure only of 40 to 25 per cent. in those subjects. Is this difference intentional? Is it necessary to maintain it? These questions may well engage the attention of the authorities and others interested.

There is also one other very important aspect of this question; an inspection of Table I will bring out the fact that the standard deviation in English is much lower than in other subjects and consequently also the frequency curve is very steep in the neighbourhood of the mode, which is very near 40 per cent. in most of the years. This means, a greater number of candidates lie in the interval 35 to 40 in the case of English than in the case of the other subjects. Hence if only the same amount of intelligence is demanded for a pass in English as in the other subjects, a good many more should be transferred to the pass list.

A proper method will be not to have an arbitrary fixed minimum, but to define the minimum to be the mark that is at a prescribed distance from the Mean, since, by assumption, the average intelligence which is measured by the Mean is constant from year to year. Now, the question is what should be this prescribed distance. In fixing this distance, the fact that the median is very close to the mean in those examinations must be taken into consideration. If we fix for the minimum, the Mean, it is implied that 50°/₀ of the candidates are to be declared unfit for a pass. Unless there is something wrong with our methods of instruction or with the nature of the candidates admitted for the courses of instruction, is it reasonable, or is it necessary to provide, in fixing the minimum, for 50°/₀ of the candidates to be declared as failures? Fixing the minimum as 5°/₀ above the average (or some such thing) will make the position still worse. This is a problem that requires careful consideration and full discussion. It is also highly controversial inasmuch as the prescribed minimum and the average mark may interact upon each other. Further, this method may introduce the danger of the backward and inefficient institutions pulling down the average, unless care is taken to see that only deserving candidates are sent up for the public examination*. Hence, it is necessary to see that the attendance certificates are made something real and this could be effected by inspecting the proportion of passes from different institu-

* It is meant that whatever may be the method adopted for the fixing and the adjustment of the minimum, the procedure should not be made known to the candidates but should be a confidential instruction to the moderating body.

tions and calling for the question papers and the class marks of the worst institutions. There is just one more remark that may be considered pertinent in this connection. The examinations of the type considered in this paper are not competitive examinations, intended to select the very best candidates, but are merely the means to give a liable passport to those that have *profitably* undergone a course of instruction, as a token of their possessing a certain minimum ability and knowledge.

IV. CORRELATION.

Next we pass on to the subject of correlation.

The following table gives the correlation co-efficients between the attainments of the candidates in the various subjects :—

TABLE III.

Year.	English | Maths.	English | Physics.	English | Chem.	English | Hist.	Maths. | Physics.	Maths. | Chem.	Physics | Chem.
1	·4188	·5262		·5349	·7910		·7220
2	·3251	·3927	·3811	·4569	·5603	·5316	·5409
3	·2584	·3637	·3970	·3563	·6284	·5789	·6085
4	·2291	·3342	·4679	·4630	·5327	·4777	·5918
5	·3870	·3950	·4339	·3983	·4912	·5447	·6098
6	·3734	·4527	·4914	·4763	·5886		·6433
Average.	*·3320*	*·4108*	*·4343*	*·4476*	*·5967*	*·5332*	*·6194*

They are shown graphically in Fig. IV.

V. DISCUSSION OF THE RESULTS.

Figure IV brings out prominently that the correlation is the greatest and most stable between the attainments of the candidates in Physics and Chemistry. It is the lowest and least stable in the case of English and Mathematics. In general the Science subjects as may be expected are found to be better correlated with one another than with English. Of all the non-language subjects, History shows the greatest correlation with English. The average values of the correlation co-efficients of the various subjects with English may be taken to give a quantitative measure of the

extent to which the power of expression in good English is necessary in the various subjects. Those measures will be more reliable, if the correl a tion between the marks in the various subjects and the marks in English Composition alone are found. It will be of great pedagogic interest to have these measures calculated.

The graphs bring out prominently the fluctuations in the correlation co-efficients. The fluctuations are not due merely to random sampling. For, the probable error in the correlation co-efficient for a population of the size considered cannot exceed ·03 (as is easily determined by the abec on page 19 of Karl Pearson's Tables. But the actual deviations are considerably beyond these limits. Hence, we have to conclude that they are real fluctuations. It is rather difficult to explain these rather large fluctuations. It is worth investigating the reason and the significance. Again in the first three years, there has been a systematic fall in almost all the correlation co-efficients. The correlations of Mathematics, Physics and History with English have undergone a *similar* decrease in these three years. Are they not significant? What do they indicate?

Again in the first year, the correlation seems to have been the greatest in all the cases. What is this due to? A satisfactory answer to these questions requires the calculation of the correlation co-efficients for some more years prior to the period considered which calculation we have not yet done.

These are some of the problems suggested by the table of correlation co-efficients and they require further study extended to the records of a larger number of years.

In conclusion, we whole-heartedly endorse the following recommendations of the Calcutta University Commission :—" A Special Examinations Board should be set up in the University. In order to deal with the problems of examinations on a large scale, it will need the services of a skilled statistician, acquainted with the modern methods of statistics, who should be a member of the Board. For, such a Board should also suggest new departures and improvements in dealing with the various problems of the examination sytsem : such matters as the problems involved in the setting of questions, of co-ordinating the marking of a set of assistant examiners, and of estimating the unavoidable element of chance in the assignment of marks, &c." *

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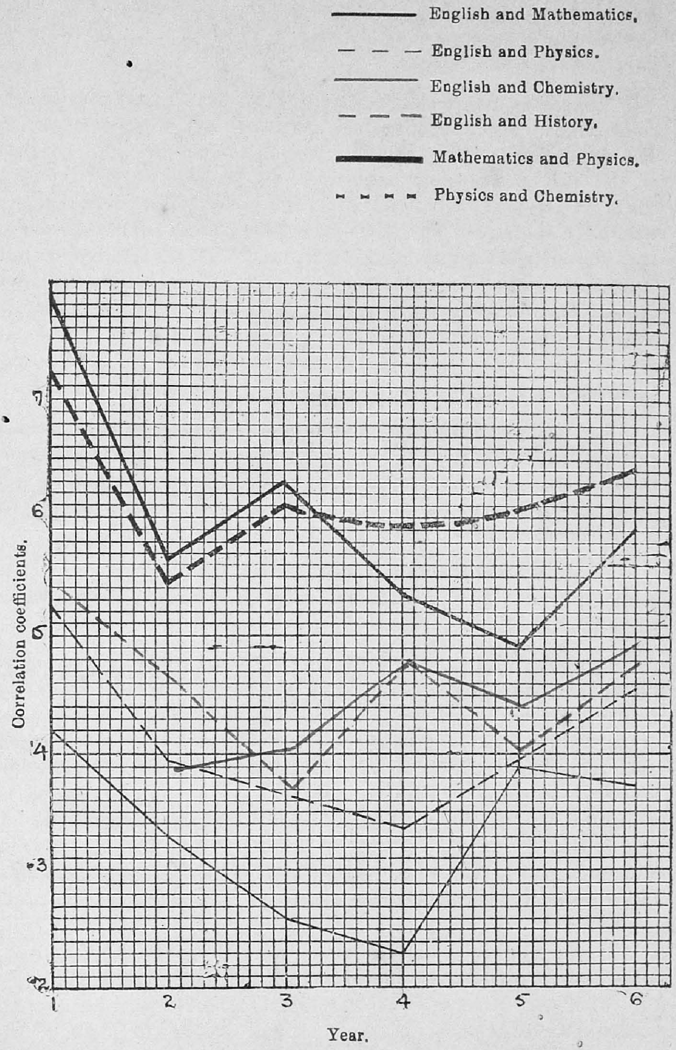
* Report of the Calcutta University Commission, Vol. V, Chapter XL.

Fig. IV.

FLUCTUATIONS IN THE CORRELATION COEFFICIENTS.



DETERMINANTS WHOSE ELEMENTS ARE EULERIAN, PREPARED BERNOULLIAN, AND OTHER NUMBERS

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[The object of this paper is to find the values of determinants of the form

$$\begin{vmatrix} a & b & c & \dots \\ b & c & d & \dots \\ c & d & e & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

known as the 'per symmetra,' the elements $a, b, c, \dots$ being Eulerian, prepared Bernoullian, and other specially defined numbers. It is believed that the values of these determinants have not been obtained by any previous writer.]

§ 1. *Notation*: We use the Σ notation throughout this paper in a particular sense as explained below.

We have $\quad \Sigma n^2 = 1^2 + 2^2 + \dots + n^2.$

Consider the series $u_{n-1} = (n-1)^2 \Sigma n^2$ which has $(n-1)$ terms.

We write

$$\Sigma u_{n-1} = \Sigma \{(n-1)^2 \Sigma n^2\} = \Sigma (n-1)^2 \Sigma (n^2),$$

omitting flower brackets.

Since $\Sigma u_{n-1} = \Sigma u_{n-2} + u_{n-1}$, we have

$$\Sigma (n-1)^2 \Sigma n^2 = (n-1)^2 \Sigma n^2 + \Sigma (n-2)^2 \Sigma (n-1)^2,$$

which is the reduction formula for our notation.

We can generalise this further in the form

$$\Sigma(a+1)^2 \Sigma(a+2)^2 \dots \Sigma(a+n)^2 = (a+1)^2 \Sigma(a+2)^2 \dots \Sigma(a+n)^2 + \Sigma a^2 \Sigma(a+1)^2 \dots \Sigma(a+\overline{n-1})^2.$$

With this notation, it is easily proved that the nth Eulerian number

$$E_n = \Sigma 1^2 \Sigma 2^2 \dots \Sigma n^2. \,*$$

In the same way, if we write

$$1 . n + 2(n+1) + \dots + r(n+r-1) = \Sigma(r, n+r-1)$$

and if $w_{r-1} = (r-1)(n+r-2) \Sigma(r, n+r-1)$.

then $\Sigma w_{r-1} = \Sigma(r-1, n+r-2) \Sigma(r, n+r-1)$

$$= (r-1)(n+r-2) \Sigma(r, n+r-1) + \Sigma(r-2, n+r-3) \Sigma(r-1, n+r-2)$$

* This formula and some interesting properties of these numbers are discussed by the authors in a paper on "Some Properties of Eulerian and Prepared Bernoudlian Numbers" communicated to the *Third Conference of the Indian Mathematical Society*, March 1921.

Of course $\sum w_{r-1}$ contains $(r-1)$ terms, and the second term above is the sum of $(r-2)$ terms, and the first is w_{r-1}. This can obviously be generalized in the form

$$\sum (a, b) \sum (a+1, b+1) \dots \sum (a+r, b+r)$$
$$= a \cdot b \sum (a+1, b+1) \dots \sum (a+r, b+r)$$
$$+ \sum (a-1, b-1) \sum (a, b) \dots \sum (a+r-1, b+r-1),$$

which is the general formula of reduction.

With this notation, it is seen that

$$b_n = 2^{2n} (2^{2n} - 1) \frac{B_n}{2_n}.$$
$$= \sum (1, 2) \sum (2, 3) \dots \sum (n-1, n)$$

where B_n = the nth Bernoullian number, that is, b_n is the prepared Bernoullian number.*

In the above reduction formulæ the integers increase as we proceed to the right.

Similar formulæ hold in the case of *descending* integers. Thus

$$\sum (n-2)^2 = 1^2 + 2^2 + \dots + (n-2)^2.$$

and if
$$v_{n-2} = n^2 \sum (n-2)^2,$$

then
$$\sum v_{n-2} = \sum \{ n^2 \sum (n-2)^2 \} = \sum n^2 \sum (n-2)^2, \text{ say.}$$

We have

$$\sum (n)^2 \sum (n-2)^2 = \sum v_{n-2} = n^2 \sum (n-2)^2 + \sum (n-1)^2 \sum (n-3)^2;$$

and so on.

The following particular cases are noteworthy :—

(i) $\sum (2n+1)^2 \sum (2n-1)^2 \dots \sum 3^2 \sum 1^2 = (2n+1)^2 (2n-1)^2 \dots 3^2 . 1^2$;

(ii) $\sum (2n)^2 \sum (2n-2)^2 \dots \sum (2^2) = (2n)^2 \sum (2n-2)^2 \sum (2n-4)^2 \dots \sum (2^2)$
$+ (2n-1)^2 (2n-3)^2 \dots (1^2)$.

§ 2. It is well-known that

$$\sec x = \sum \left\{ E_r \frac{x^{2r}}{(2r)!} \right\},$$

where E_r is the rth Eulerian number.

Differentiating

$$\sec x . \tan x = \sum \left\{ E_r \frac{x^{2r-1}}{(2r-1)!} \right\}.$$

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* This formula and some interesting properties of these numbers are discussed by the authors in a paper on "Some Properties of Eulerian and Prepared Bernoullian Numbers" communicated to the *Third Conference of the Indian Mathematical Society*, March 1921.

Generally, it can be proved by *mathematical induction*,* that

(i) $(2n)!\ \sec x.\tan^{2n} x=\Sigma\{E_{r+n}-\Sigma(2n-1)^2.\ E_{(r+n-1)}$

$$+\ \Sigma(2n-1)^2\ \Sigma(2n-3)^2\ E_{r+n-2}-+\dots\Big\}\frac{x^{2r}}{(2r)!};\ (1)$$

(ii) $(2n+1)!\ \sec x.\tan^{2n+1} x=\Sigma\{E_{r+n}-\Sigma(2n)^2.\ E_{r+n-1}+$

$$\Sigma(2n)^2\ \Sigma(2n-2)^2\ E_{r+n-2}-+\dots\Big\}\frac{x^{2r-1}}{(2r-1)!}.\ (2)$$

§ 3. We have proved in a paper on " A Table of Values of Thirty Eulerian Numbers " that the expansion of ($\sec x\tan^m x$) in ascending powers of x may be written in the form

$$\sec x\ \tan^m x=(m)!\Big\{\frac{x^m}{m!}+\frac{x^{m+2}}{m+2!}\Sigma(m+1)^2+$$

$$\frac{x^{m+4}}{m+4!}\Sigma(m+1)^2\ \Sigma(m+2)^2+\dots,\qquad (3)$$

$$+\frac{x^{m+2r}}{(m+2r)!}\Sigma(n+1)^2\ \Sigma(n+r)^2\dots\Sigma(n+r)^2+\dots\Big\}$$

when m is an integer.

§ 4. Now comparing the expansion (1) with (3), we find that the co-efficients of powers of x lower than x^{2n} must vanish in (1). If we denote by S_r the quantity $\Sigma(2n-1)^2\ \Sigma(2n-3)^2\ \dots\ \Sigma(2n-2r+1)^2$, we thus obtain the following equations :—

$$\left.\begin{array}{l} E_n\ -S_1E_{n-1}+S_2E_{n-2}-\dots+(-1)^nS_nE_0\ =0 \\ E_{n+1}-S_1E_n\ +S_2E_{n-1}-\dots+(-1)^nS_nE_1\ =0 \\ E_{n+2}-S_1E_{n+1}+S\ \ E_n\ \ -\dots+(-1)^nS_nE_2\ =0 \\ \dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots \\ E_{2n-1}-S_1E_{2n-2}+S_2E_{2n-3}-\dots+(-1)^nS_nE_{n-1}=0. \end{array}\right\}(4).$$

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* For, assuming the results for values up to $2n$ and differentiating (i), we have

$$(2n+1)!\ \sec x.\tan^{2n+1}x+2n.(2n)!\sec x.\tan^{2n-1}x$$

$$=\Sigma\{E_{r+n}-\Sigma(2n-1)^2.\ E_{r+n-1}+$$

$$\Sigma(2n-1)^2\ \Sigma(2n-3)^2.E_{r+n-2}-+\dots\Big\}\frac{x^{2r-1}}{(2r-1)!}$$

which gives (ii) for the next higher value, if we bear in mind that

$$2n.(2n)!\sec x.\tan^{2n-1}x=(2n)^2.\Sigma\{E_{r+n-1}-\Sigma(2n-2)^2\ E_{r+n-2}$$

$$+\Sigma(2n-2)^2\ \Sigma(2n-4)^2\ E_{r+n-3}-+\dots\Big\}\frac{x^{2r-1}}{(2r-1)!}$$

and $\Sigma(2n)^2\ \Sigma(2n-2)^2=(2n)^2\ \Sigma(2n-2)^2+\Sigma(2n-2)^2\ \Sigma(2n-4)^2$

&c. &c.

Similarly for the other case.

Equating the co-efficients of x^{2n} in (1) and (3),

$$E_{2n}-S_1E_{2n-1}+S_2E_{2n-2}-\dots+(-1)^nS_nE_n=(2n!)^2.\quad (5)$$

Now write

$$\Delta_{2n} = \begin{vmatrix} E_0 & E_1 & \dots & E_n \\ E_1 & E_2 & \dots & E_{n+1} \\ E_2 & E_3 & \dots & E_{n+2} \\ \dots & \dots & \dots & \dots \\ E_n & E_{n+1} & \dots & E_{2n} \end{vmatrix}$$

The above equations easily give (after a little reduction and re-arrangement), the relation

$$\Delta_{2n} = (2n\,!)^2 \,.\, \Delta_{2n-2},$$

whence

$$\Delta_{2n} = (2n\,!)^2\,(2n-2\,!)^2\,.\,(2n-4\,!)^2\,..\,(2\,!)^2$$

$$= \{\,(1.2)^n\,(3.4)^{n-1}\,(5.6)^{n-2}\,..(2n-1.2n)\,\}^2, \qquad (6)$$

Example.

$$\begin{vmatrix} E_0 & E_1 & E_2 \\ E_1 & E_2 & E_3 \\ E_2 & E_3 & E_4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 5 \\ 1 & 5 & 61 \\ 5 & 61 & 1385 \end{vmatrix} = 2304. = (4\,!)^2\,(2\,!)^2.$$

Also, solving for S_1 from (4) and (5),

$$-S_1\,\Delta_{2n} = \begin{vmatrix} E_n & E_{n-2} & E_{n-3} & \dots & E_0 & 0 \\ E_{n+1} & E_{n-1} & E_{n-2} & \dots & E_1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ E_{2n} & E_{2n-2} & E_{2n-3} & \dots & E_n & -(2n\,!)^2 \end{vmatrix}$$

$$\therefore\ S_1\,\Delta_{2n} = (2n\,!)^2.\begin{vmatrix} E_0 & E_1 & \dots & E_{n-1} \\ E_1 & E_2 & \dots & E_n \\ \dots & \dots & \dots & \dots \\ E_{n-2} & E_{n-1} & \dots & E_{2n-3} \\ E_n & E_{n+1} & \dots & E_{2n-1} \end{vmatrix} \qquad (7)$$

[Notice that the row $|\,E_{n-1}, E_n\ \dots\ E_{2n-2}\,|$ is absent in the above.]

We thus obtain the result

$$\begin{vmatrix} E_0 & E_1 & E_2 & \dots & E_{n-1} \\ E_1 & E_2 & E_3 & \dots & E_n \\ \dots & \dots & \dots & \dots & \dots \\ E_{n-2} & E_{n-1} & E_n & \dots & E_{2n-3} \\ E_n & E_{n+1} & E_{n+2} & \dots & E_{2n-1} \end{vmatrix} = \frac{2n\,(2n-1)\,(4n-1)}{6} \times (2n-2\,!)^2\,(2n-4\,!)^2 \dots\dots (2\,!)^2. \qquad (8)$$

Ex. If $m=3$, we get

$$\begin{vmatrix} E_0 & E_1 & E_2 \\ E_1 & E_2 & E_3 \\ E_3 & E_4 & E_5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 5 \\ 1 & 5 & 61 \\ 61 & 1385. & 50521 \end{vmatrix} = 126720.$$

Solving generally for S_r, we have

$$\begin{vmatrix} E_0 & E_1 & \dots & E_{n-r-1} & E_{n-r+1} & \dots & E_n \\ E_1 & E_2 & \dots & E_{n-r} & E_{n-r+2} & \dots & E_{n+1} \\ \dots & \dots & & \dots & & \dots & \\ E_{n-1} & E_n & \dots & E_{2n-r-2} & E_{2n-r} & \dots & E_{2n-1} \end{vmatrix} = (2n-2\,!)^2\,(2n-4\,!)^2 \dots (2\,!)^2\,S_r. \qquad (9)$$

Since S_r can be readily calculated when r is given, we thus obtain the values of such determinants for different r. Thus writing $n=3$, $r=2$, we have $S_r = 439$,

and
$$\begin{vmatrix} E_0 & E_2 & E_3 \\ E_1 & E_3 & E_4 \\ E_2 & E_4 & E_5 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 61 \\ 1 & 61 & 1385 \\ 5 & 1385 & 50521 \end{vmatrix}$$
$$= 1,011,456$$
$$= (24)^2 . (2)^2 . 439.$$

One case of great interest is that obtained by putting $r = n$; then
$$S_n = \Sigma(2n-1)^2\ \Sigma(2n-3)^2 \dots \Sigma 1^2$$
$$= (2n-1)^2 (2n-3^2) (2n-5)^2 \dots 1^2;$$

and we obtain the equation
$$\begin{vmatrix} E_1 & E_2 & \dots & E_n \\ E_2 & E_3 & \dots & E_{n+1} \\ \dots & \dots & \dots & \dots \\ E_n & E_{n+1} \dots & & E_{2n-1} \end{vmatrix} = (2!)^2 (4!)^2 \dots (2n-2!)^2 \dots \quad 1^2 . 3^2 . 5^2 \dots (2n-1)^2. \quad (10)$$
$$= (2n-1!)^2 (2n-3!)^2 \dots (3!)^2 (1!)^2$$
$$= [1^n . (2 . 3)^{n-1} (4 . 5)^{n-2} \dots (2n-2 . 2n-1)^1]^2.$$

Ex. If $n = 3$, we have
$$\begin{vmatrix} 1 & 5 & 61 \\ 5 & 61 & 1385 \\ 61 & 1385 & 50521 \end{vmatrix} = 518,400 = 2^2 . 24^2 . 3^2 . 5^2.$$

§ 5. Equating the co-efficients of x^{2n+2s}, we obtain from (1) and (3)
$$E_{2n+s} - \Sigma(2n-1)^2 E_{2n+s-1} +$$
$$+ (-1)^n \Sigma(2n-1)^2 \Sigma(2n-3)^2 \dots \Sigma(1^2) E_{n+s}$$
$$= (2n!)^2 \Sigma(2n+1)^2 \Sigma(2n+2)^2 \dots \Sigma(2n+s)^2. \quad (11)$$

Hence from (4) and (11), we obtain,
$$\begin{vmatrix} E_0 & E_1 & \dots & E_n \\ E_1 & E_2 & \dots & E_{n+1} \\ \dots & \dots & \dots & \dots \\ E_{n-1} & E_n & \dots & E_{2n-1} \\ E_{n+s} & E_{n+s+1} & \dots & E_{2n+s} \end{vmatrix} = (2n!)^2 \Sigma(2n+1)^2 \Sigma(2n+2)^2 \dots \Sigma(2n+s)^2 . \quad \Delta_{2n-2}$$
$$= (2n!)^2 (2n-2!)^2 \dots (2!)^2 \Sigma(2n+1)^2 \Sigma(2n+2)^2 \dots \Sigma(2n+s)^2. \quad (12)$$

Ex. Writing $n=s=2$, we have
$$\begin{vmatrix} E_0 & E_1 & E_2 \\ E_1 & E_2 & E_3 \\ E_4 & E_5 & E_6 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 5 \\ 1 & 5 & 61 \\ 1385 & 50521 & 2702765 \end{vmatrix}$$
$$= 8,031,744.$$
$$= 2^2 . (24)^2 . 3486.$$
$$= 2^2 . (24)^2 . (\Sigma 5^2 \Sigma 6^2).$$

Writing $n=2$, $5=3$, we get

$$\begin{vmatrix} E_0 & E_1 & E_2 \\ E_1 & E_2 & E_3 \\ E_5 & E_6 & E_7 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 5 \\ 1 & 1 & 61 \\ 50,521 & 2,702,765 & 199,360,981 \end{vmatrix}$$

$$= 647,907,840.$$
$$= 2^2.(24)^2\ (281,210)$$
$$= 2^2.(24)^2\ (\Sigma 5^2\ \Sigma 6^2\ \Sigma 7^2)$$

§ 6. Now considering (2) and comparing it with (3), for m odd, we obtain the following equations:—

$$\left.\begin{aligned} E_{n+1} - k_1 . E_n + k_2 E_{n-1} - \ldots + (-1)^n k_n E_1 &= 0 \\ E_{n+2} - k_1 . E_{n+1} + k_2 E_n - \ldots + (-1)^n k_n E_2 &= 0 \\ E_{2n} - k_1 . E_{2n-1} + k_2 E_{2n-2} - \ldots + (-1)^n k_n E_n &= 0 \end{aligned}\right\} \qquad (4 . a)$$

$$E_{2n+1} - k_1 E_{2n} + k_2 E_{2n-1} - \ldots + (-1)^n k_n E_{n+1} = (2n+1\,!)^2. \quad (5 . a)$$

where $k_r = \Sigma(2n)^2\ \Sigma(2n-2)^2 . \Sigma(2n-2r+2)^2$.

Without going into details, we may state in particular that we obtain (10) and also the persymmetric,

$$\begin{vmatrix} E_2 & E_3 & \ldots & E_{n+1} \\ E_3 & E_4 & \ldots & E_{n+2} \\ \ldots & \ldots & \ldots & \ldots \\ E_{n+1} & E_{n+2} & \ldots & E_{2n} \end{vmatrix} = [1^n\,(2.3)^{n-1}\ \ldots\ (2n-2\,.\,2n-1)^1]^2. \times \Sigma(2n)^2\ \Sigma(2n-2)^2 \ldots \Sigma 2^2. \quad (10 . a)$$

Ex. $\begin{vmatrix} E_2 & E_3 \\ E_3 & E_4 \end{vmatrix} = \begin{vmatrix} 5 & 61 \\ 61 & 1385 \end{vmatrix} = 3204 = 6^2 . (89)$

§ 7. We next proceed to evaluate in a similar manner the values of determinants involving the prepared Bernoullian numbers.

We shall write

$$b_n = 2^{2n}(2^{2n}-1)\frac{B_n}{2n}.$$

It is well-known that

$$\tan x = \Sigma\, b_n \frac{x^{2n-1}}{2n-1\,!}.$$

Differentiating and transposing

$$\tan^2 x = \sum_1^{\infty} b_n \frac{x^{2n-2}}{2n-2\,!}.$$

Differentiating again and using the above

$$\therefore \quad 2\,!\tan^3 x = \Sigma(b_{n+1} - 1.2b_n)\frac{x^{2n-1}}{2n-1\,!}.$$

Proceeding in this manner, we can write down, by induction, that,

$$(2n-2)!\tan^{2n-1} x = \Sigma \frac{x^{2r-1}}{2r-1\,!}\Big\{b_{r+n-1} - L_1 b_{r+n-2} + L_2 b_{r+n-3} - \ldots + (-1)^{n-1} L_{n-1} b_r\Big\} \ldots \quad (13)$$

where $L_r = \Sigma(2n-3, 2n-2)\, \Sigma(2n-5, 2n-4) \ldots \Sigma(2n-2r-1, 2n-2r)$.

$$(2n-1)!\ \tan^{2n} x = \Sigma \frac{x^{2r}}{2r!} \left\{ b_{r+n} - T_1 b_{r+n-1} + T_2 b_{r+n-2} \ldots\ldots + (-1)^{n-1} T_{n-1} b_{r+1} \right\} \quad (14)$$

where $T_r = \Sigma(2n-2, 2n-1)\, \Sigma(2n-4, 2n-3) \ldots \Sigma(2n-2r, 2n-2r+1)$

§ 8. From (13), we obtain the following equations :—

$$\left.\begin{array}{l} b_n - L_1 b_{n-1} + L_2 b_{n-2} - \ldots + (-1)^{n-1} L_{n-1} b_1 = 0 \\ b_{n+1} - L_1 b_n + L_2 b_{n-1} - \ldots + (-1)^{n-1} L_{n-1} b_2 = 0 \\ b_{n+2} - L_1 b_{n+1} + L_2 b_n - \ldots + (-1)^{n-1} L_{n-1} b_3 = 0 \\ \ldots \\ b_{2n-2} - L_1 b_{2n-3} + L_2 b_{2n-4} - \ldots + (-1)^{n-1} L_{n-1} b_{n-1} = 0 \\ b_{2n-1} - L_1 b_{2n-2} + L_2 b_{2n-3} - \ldots + (-1)^{n-1} L_{n-1} b_n \\ \qquad = (2n-1)!\,(2n-2)! \end{array}\right\} \quad (15)$$

Hence, we obtain

$$\begin{vmatrix} b_1 & b_2 & b_3 & \ldots & b_n \\ b_2 & b_3 & b_4 & \ldots & b_{n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ b_n & b_{n+1} & \ldots & \ldots & b_{2n-1} \end{vmatrix} = 2n-1!\ 2n-2! \begin{vmatrix} b_1 & b_2 & \ldots & b_{n-1} \\ b_2 & b_3 & \ldots & b_n \\ \ldots & \ldots & \ldots & \ldots \\ b_{n-1} & b_n & \ldots & b_{2n-3} \end{vmatrix}$$

$$= 2n-1!\ 2n-2!\ 2n-3!\ 2n-4! \ldots 2!\ 1!. \quad (16)$$

Ex.
$$\begin{vmatrix} b_1 & b_2 & b_3 \\ b_2 & b_3 & b_4 \\ b_3 & b_4 & b_5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 16 \\ 2 & 16 & 272 \\ 16 & 272 & 7936 \end{vmatrix}$$

$$= 34,560$$

$$= 1.\ 2.\ 6.\ 24.\ 120.$$

§ 9. Solving for Lr in general, we have,

$$\begin{vmatrix} b_1 & b_2 & \ldots & b_{n-r-1} & b_{n-r+1} & \ldots & b_n \\ b_2 & b_3 & \ldots & b_{n-r} & b_{n-r+2} & \ldots & b_{n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ b_{n-1} & b_n & \ldots & b_{2n-r-3} & b_{2n-r-1} & \ldots & b_{2n-2} \end{vmatrix} 2n-1!\ 2n-2$$

$$= Lr.\ 2n-1!\ 2n-2! \ldots\ 2!\ 1!.$$

Thus

$$\begin{vmatrix} b_1 & b_2 & \ldots & b_{n-r-1} & b_{n-r+1} & \ldots & b_n \\ b_2 & b_3 & \ldots & b_{n-r} & b_{n-r+2} & \ldots & b_{n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ b_{n-1} & b_n & \ldots & b_{2n-r-3} & b_{2n-r-1} & \ldots & b_{2n-2} \end{vmatrix}.$$

$$= Lr\ (2n-3)!\ (2n-4)! \ldots 2!\ 1!. \quad (17)$$

In particular, writing $r = 1$,

$$\begin{vmatrix} b_1 & b_2 & \dots & b_{n-2} & b_n \\ b_2 & b_3 & \dots & b_{n-1} & b_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ b_{n-1} & b_n & \dots & b_{2n-4} & b_{2n-2} \end{vmatrix}$$

$$= 2n-3\,!\,2n-4\,!\,2n-5\,!\,\dots\,1\,! \times L_1.$$

and $L_1 = 1\,.\,2 + 2\,.\,3 + \dots + (2n-3)(2n-2)$
$= (2n-3)(2n-2)(2n-1)/3.$

$$\therefore\quad D = \frac{(2n-3)(2n-2)(2n-1)}{3}(2n-3)!\,(2n-4)!\dots 1\,!. \qquad (18)$$

§ 10. From (14), we obtain the following equations—

$$\left.\begin{array}{l} b_{n+1} - T_1 b_n + T_2 b_{n-1} - \dots + (-1)^{n-1} T_{n-1} b_2 = 0 \\ b_{n+2} - T_1 b_{n+1} + T_2 b_n - \dots + (-1)^{n-1} T_{n-1} b_3 = 0 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ b_{2n-1} - T_1 b_{2n-2} + T_2 b_{2n-3} - \dots + (-1)^{n-1} T_{n-1} b_n = 0 \end{array}\right\} \qquad (19)$$

$$b_{2n} - T_1 b_{2n-1} + T_2 b_{2n-2} - \dots + (-1)^{n-1} T_{n-1} b_{n+1}$$
$$= (2n)!\,(2n-1)!\,. \qquad \dots (20)$$

Hence as before,

$$\begin{vmatrix} b_2 & b_3 & \dots & b_{n+1} \\ b_3 & b_4 & \dots & b_{n+2} \\ \dots & \dots & \dots & \dots \\ b_{n+1} & b_{n+2} & \dots & b_{2n} \end{vmatrix} = 2n\,!\,(2n-1)!\,(2n-2)!\,\dots\,2\,!\,1\,! \quad (21)$$

and

$$\begin{vmatrix} b_2 & b_3 & \dots & b_{n-r} & b_{n-r+2} & \dots & b_{n+1} \\ b_3 & b_4 & \dots & b_{n-r+1} & b_{n-r+3} & \dots & b_{n+2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n+1} & b_{n+2} & \dots & b_{2n-r-1} & b_{2n-r+1} & \dots & b_{2n-1} \end{vmatrix}$$
$$= T_r.\,(2n-2)!\,(2n-3)!\,..\,1\,! \quad \dots \quad (22)$$

§ 11. Similarly we obtain, from (19), by solving for T_{n-1}, that,

$$\begin{vmatrix} b_3 & b_4 & \dots & b_{n+1} \\ b_4 & b_5 & \dots & b_{n+2} \\ \dots & \dots & \dots & \dots \\ b_{n+1} & b_{n+2} & \dots & b_{2n-1} \end{vmatrix} = T_{n-1} \begin{vmatrix} b_2 & b_3 & \dots & b_n \\ b_3 & b_4 & \dots & b_{n+1} \\ \dots & \dots & \dots & \dots \\ b_n & b_{n+1} & \dots & b_{2n-2} \end{vmatrix}$$

$$= T_{n-1}\,.\,2n-2\,!\,2n-3\,!\,\,..\,\,2\,!\,.\,1\,!.$$

[*To be continued.*]

MATHEMATICS IN INDIA—THEN AND NOW.

By V. Sankaran.

(*Continued from page 224, Vol. XIII*).

Let us next consider the Tamil work *Kanakkadikaram*. Its contents are roughly :—(1) notation, (2) tables of weights, (3) relative weights and sizes, (4) measurement of time, (5) alloys, (6) lengths, (7) table of capacity, (8) tables of time and life of different animals, (9) areas, (10) military units, (11) mensuration, (12) time and work, (13) partnership, (14) mixtures, and (15) miscellaneous examples. Only such problems are dealt with as are likely to be useful in the practical application to other sciences and to life. The style is popular, and there are few technical phrases employed in the exposition of the subject-matter.

We shall examine these contents at some length : The fractions $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{20}$, $\frac{1}{80}$, $\frac{1}{320}$ have special names—*arai*, *kâl*, *araikâl*, *mâ*, *kâni*, and *mundirai*. Any given fraction is usually expressed in terms of these and the approximation is taken to the nearest mundirai. $\frac{1}{320^2}$ was termed the *kīlmundirai* ; there was the fraction *immi* equal to $\frac{2}{21}$ of a *kīlmundirai*. This shows that, where necessary, people used to calculate $\frac{2}{21} \times \frac{1}{320} \times \frac{1}{320}$ of a unit. However, they generally stopped with the nearest *mundirai*. Sub-multiples of *immi* are also mentioned.

In connection with the tables of weights and measures, we find the following table for gold :—

2 grains	= 1 pilavu.	
2 pilavus	= 1 kundri.	
2 kundris	= 1 manjadi.	
20 manjadis	= 1 kalanji	(= $\frac{2}{3}$ sovereign).
2 kalanjis	= 1 kaisa.	
4 kaisas	= 1 palam.	
100 palams	= 1 nirai.	
2 nirais	= 1 tulam.	
32 tulams	= 1 baram.	

This system is still in vogue in parts of the Tinnevelly District. Particulars about the weights of well-known substances are added and are followed by other interesting approximations. Thus we are told that

one measure of	clay	weighs	17	palams
,,	sand	,,	20	,,
,,	paddy	,,	6	,,
,,	rice	,,	10	,,
,,	salt	,,	16	,,

Again we read that

one measure of	til	contains	195200	seeds.
,,	paddy	,,	14400	,,
,,	rice	,,	39800	,,
,,	peas (green)	,,	14800	,,
,,	beans	,,	1800	,,
,,	pepper	,,	12800	,,

Further we learn that clear water weighs 12 palams per measure, ordinary water 13 palams, and sea water 17 palams.

In Section (4), the author begins with the description of a water clock. According to him, a cylindrical basin 12 fingers in diameter, and 5 in height will be emptied through a small hole at the bottom made with a nail of particular size at the rate of a finger length per *naligai*.*

Section (5) begins with the composition of the alloys in common use : 8 parts by weight of copper and 2 of tin gives bronze ; $7\frac{1}{2}$ of copper and 3 of zinc gives brass and so on.

In (6), we have the following table of lengths :—

8 anūs	=	1 kadireludu.
8 kadireludus	=	1 panchitru.
8 panchitrus	=	1 mayirmunai.
8 mayirmunais	=	1 sirumanal.
8 sirumanals	=	1 sirukadugu.
8 sirukadugus	=	1 ellu.
8 ellus	=	1 arisi.
8 arisis	=	1 viral (finger).
12 virals	=	1 s'an.

This gives an indication of the high degree of accuracy attained in the measurement of length. That measurements to this degree of accuracy were made, and that there were appliances to make them are matters of fact. We find in old medical books descriptions of very fine surgical instruments, even for the purpose of splitting the hair in two.

We next meet with the following table of measures :—

5 sevidus	=	1 olockku (ஆழாக்கு).
2 olockkus	=	1 ulakku (உழக்கு).
4 ulakkus	=	1 measure.
8 measures	=	1 kuruni.
4 kurunis	=	1 tuni.
3 tunis	=	1 kalam.

* The standards were reserved in the King's palace.

The first and the last but one are out of use; the others are still used as units for measuring paddy in the Tamil districts.

(8) gives the table of time measure as follows :—

2 kannimais	=	1 nodi.
2 nodis	=	1 mallirai.
2 mallirais	=	1 guru.
2 gurus	=	1 uyir.
6 uyirs	=	1 kshanikam.
12 kshanikams	=	1 vinadi (24″).
60 vinadis	=	1 naligai.
$7\frac{1}{2}$ naligais	=	1 jamam.
8 jamams	=	1 day.
30 days	=	1 month.
12 months	=	1 year.
		...... etc.

We are given also the duration of the life of some animals. Thus, the life of man or elephant is 100 years; that of cow or bullock is 20; of buffalo 30; of camel 73.

The following are given to be the military units :—1 padâti consists of 10 elephants, 3 chariots, 100 cavalry and 1000 infantry.

82 padâtis	=	1 tandu.
100 tandus	=	1 akshohini.

Here is an estimate of private property :—

25 elephants	=	1 ani.
85 horses	=	1 mottam.
1 divam.	=	800 sheep, 80 bulls or cows and 80 buffaloes

Coming to Section (11), we find the following formulæ :—

(i) the area of a trapezoidal field is given as $\left(\frac{a+b}{2}\right) d$, where a, b are the parallel sides and d is the distance between them; (ii) the area of a circle is $(\frac{1}{2}c)(\frac{1}{2}d)$, where c is the circumference and d is the diameter, (iii) circumference is $\frac{2}{5}(8d)$, (iv) the area of a triangle is half the base into the altitude, &c.

Later on, problems on the number of seedlings that can be planted in a unit area and the amount of paddy harvested are practically solved by counting the grains in plants selected at four corners and centre. He gives also the ratio between the quantity of paddy, and rice that can be obtained from it as 16 to 7.

In Section (13), we have examples on proportion, including questions on gold and gold qualities.

Section (15), really deals with novel things. According to the author, the number of nuts in a jack fruit is to the number of thorns at the base of the stem as *six* to *five*. In the case of a pumpkin, he says,

> " கீற்றெண்ணி முத்திற்று கீழாறினல் மாறி
> வேற்றை யஞ்சதனில் மிகப்பெருகாப்—பார்த்ததிலே
> பாதிதள்ளி முன்னிற் பகிரவி தியாகும்
> பூசினிக்காய் தோறும்புகல்."

which means that each section contains 135 seeds. This is followed by an interesting example.

" A milkman who usually took milk to a king was one day waylaid by the gate-keepers, 24 in number. Each took one measure of milk and replaced the same by a measure of water. The king finding the milk tasteless fined the milkman Rs. 24, on which the real culprits were pointed out. Required the share of each gatekeeper in the fine."

Here is another :—

" Given three vessels of capacity 10, 7, 3 with the biggest full of liquid, show how to divide, with these vessels only, the liquid into two equal parts."

There are also problems leading to simple equations and problems on gold alloys.

This short account makes it evident that people in early times in India based their teaching on definite educational priciples, *viz.*: the correlation of mathematics to physics, and that of both to life. No question in mathematics stripped of its settings and utility, was discussed,

The knowledge was practical, though they had not a laboratory and conveniences to carry on elaborate experiments ; and whatever was taught to the student was useful to him in life. The problems were not the dry questions of modern text-books. Most of them related to the actual surroundings of the student.

An attempt has been made in the preceding pages to show that very useful knowledge was imparted to the student before the advent of western education. The authors from whom we have quoted make it clear that more elaborate works were in existence in their times. But of those other works we know very little. It must be our ambition to unearth as many of the earlier works in mathematics as possible. Very little in fact is known about Sanskrit works by authors like Bhaskara, Mahavira, etc. I appeal to all interested in the study of mathematics to try each in his own way to find out and study as many works as he can discover, and build up a real history of Indian Mathematics.*

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* The reader is referred to the Proceedings of the Benares Mathematical Society, Vol. II, Part II, for an elaborate account of problems in higher mathematics discussed by Bhaskara and Kamalakara.

SHORT NOTES:

Some Criteria for Divisibility.

1. Let N stand for the number to be tested,

N' „ for the same with the units' digit removed,
a „ for the units' digit in N,
d „ for the divisor (with two digits, say),
b „ for the digit in the ten's place in d.

Then the following simple Theorems are easily proved :—

Theorem I. For all divisors d ending in unity,

$$(N' \sim ab) \equiv 0 \text{ (mod. } d)$$

Proof: $N = a + 10\,N'$, $d = 1 + 10b$.

$$\therefore \qquad (N' - ab) = \frac{N - a}{10} - ab = \frac{N - ad}{10}.$$

Hence, if N is divisible by d, $(N' - ab)$ is also divisible by d, which proves the Theorem.

Theorem II. For all divisors d ending in 3,

$$N' + (3b + 1)\,a \equiv 0 \text{ (mod. } d)$$

Proof: $$N' + (3b + 1)\,a = \frac{N - a}{10} + (3b + 1)\,a$$

$$= \frac{N + 3ad}{10},$$

whence the result follows.

Theorem III. For all divisors d ending in 7,

$$N' - (3b + 2)\,a \equiv 0 \text{ (mod. } d).$$

Proof as before.

Theorem IV. For all divisors ending in 9,

$$N' + (b + 1)\,a \equiv 0 \text{ (mod. } d)$$

Proof as before.

[*N.B.*—Cases where d ends in 2, 4, 5, 6, 8 are easily reduced to the above.]

2. The utility of these Theorems may be illustrated by a few examples:

Ex. 1.—Test for divisibility by 31 the number 107229.

Here $N' = 10722$, $a = 9$, $b = 3$, in Th. I.

$$\therefore \qquad (N' - 3a) \equiv 10722 - 27 = 10695$$

which should be similarly tested.

Applying the same Theorem again, we get

$$1069 - 15 = 1054,$$

which has to be tried.

Repeating the process

$$105 - 12 = 93.$$
$$9 - 9 = 0,$$

which proves the divisibility of N by 31.

Ex. 2.—Apply Theorem II to number 81786, the divisor being 43.

Here the formula is

$$(N' + 13a) \equiv 0 \text{ (mod. } 43);$$

and we arrange the work compactly as under:

8178	6	78
825	6	78
90	3	39
12	9	

which proves the result.

Ex. 3. $N = 1937$, $d = 149$. Here Theorem IV may be used.

Thus

193	7	105
29	8	120
14	9	

PRASANNA K. DAS.

Perpetual Calendar for Every Year of the Christian Era

Designed by C. GANAPATI AIYAR, B A., L.T.,
First Assistant, Madura College, Madura.

Directions for use,

1. In both styles, every year whose number is divisible by 4 is a Leap Year and is shown in red ink ; but in the New Style 1700, 1800 and 1900 are *not* leap years.

[NOTE.—In *Europe* the last day of the Old Style was October 4, 1582 (Thursday) and the next day was declared the first day of the New Style October 15, 1582 (Friday).

In *England* the last day of the Old Style was September 2, 1752 (Wednesday) and the next day was declared the first day of the New Style, September 14, 1752 (Thursday)].

2. From the Day of the Week below any given Month count as many steps to the *right* or *left* as the column containing the number of a given Year is to the *right* or *left* of the column containing the Century number of the Year ; you will have the Week-Day of the 1st of that Month.

To find the Week-day on the 1st of any Month of a Year :

The other days of the month can then be readily calculated.

[Example 1. *To find the week-day of January 1st, 1875.*

Here the century number is 18 and the year number is 75 and both are in the same column. Therefore January 1st, 1875 was Friday.

Example 2. *The battle of Waterloo was fought on June 18th, 1815. What was the week-day of the battle?*

The column containing the year number 15 is 2 steps to the right of the column containing the century number 18. Therefore June 1st, 1815 fell on Thursday and consequently June 18th was Sunday.]

Century Numbers. Old Style.	1 8 15	2 9 16	3 10 17	4 11 18	5 12 19	6 13 20	0 7 14 21	etc.
Century Numbers. New Style.	— —	— 18	— —	15 19	16 20	— —	17 21	etc.
Year Numbers.	01	02	03	—	04	05	06	
	07	—	08	09	10	11	—	
	12	13	14	15	—	16	17	
	18	19	—	20	21	22	23	
	—	24	25	26	27	—	28	
	29	30	31	—	32	33	34	
	35	—	36	37	38	39	—	
	40	41	42	43	—	44	45	
	46	47	—	48	49	50	51	
	—	52	53	54	55	—	56	
	57	58	59	—	60	61	62	
	63	—	64	65	66	67	—	
	68	69	70	71	—	72	73	
	74	75	—	76	77	78	79	
	—	80	81	82	83	—	84	
	85	86	87	—	88	89	90	
	91	—	92	93	94	95	—	
	96	97	98	99	—	—	00	
Months.	Jan. Oct. —	— May. —	L. Feb. Aug. —	Feb. Mar. Nov.	— Jun. —	— Sep. Dec.	L. Jan. Apr. July	
Week-Days.	*F.*	*S.*	S.	*M.*	*T.*	*W.*	*Th.*	

SOLUTIONS.

Questions 565 & 730.

(S. KRISHNASWAMI IYENGAR):—Shew that the locus of the orthopoles of tangents to the maximum inscribed ellipse of a triangle is a straight line through the ortho-centre.

Remarks by M. Bhimasena Rao.

Mr. N. Sankara Aiyar, M.A., points out that the question is incorrect by showing that the orthopoles of two special tangents to the maximum inscribed ellipse, *viz.*, Feuerbach's tangents, are not collinear with the ortho-centre. This will be evident from Question 328 in the Journal for October 1911, which states that the envelope of a line whose orthopole lies on a given line is a parabola.

The co-ordinates of the orthopole of $p\alpha + q\beta + r\gamma = 0$ are $(-p+q\cos C+r\cos B)$ $(-p\cos B\cos C + q\cos B + r\cos C)$, etc, from Question 328 cited above. If $p\alpha + q\beta + r\gamma = 0$ touches the maximum inscribed ellipse,

$$\frac{a}{p} + \frac{b}{q} + \frac{c}{r} = 0.$$

The correct locus is obtained by eliminating p, q, r from this equation and from

$$\frac{(-p+q\cos C + r\cos B)(-p\cos B\cos C + q\cos B + r\cos C)}{\alpha} =$$

$$\frac{\dots\dots}{\beta} = \frac{\dots\dots}{\gamma}.$$

The following equations giving the co-ordinates of the orthopole of $p\alpha + q\beta + r\gamma = 0$ seem to be best suited to determine the locus of orthopoles of tangents to any inconic of the triangle of reference :—

$$\left(\frac{b}{q} - \frac{c}{r}\right)\cos A.\,\alpha + \left(\frac{c}{r} - \frac{a}{p}\right)\cos B.\,\beta + \dots\dots = 0 \qquad (1)$$

is the equation of the directrix of the parabola touching $p\alpha + q\beta + r\gamma = 0$ and the sides of the triangle ;

and

$$\frac{\left(\frac{q}{b} - \frac{r}{c}\right)\alpha}{-p + q\cos C + r\cos B} + \dots + \dots = 0 \qquad \dots \qquad (2)$$

is the equation of the pedal line of the point at infinity on $p\alpha + q\beta + r\gamma = 0$.

If $p\alpha + q\beta + r\gamma = 0$ touches the inconic $(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} = 0$. We have

$$\frac{l}{p} + \frac{m}{q} + \frac{n}{r} = 0. \qquad \dots \qquad \dots \qquad (3)$$

Writing $\beta \cos B - \gamma \cos C = S_1$, $\gamma \cos C - \alpha \cos A = S_2$, $\alpha \cos A - \beta \cos B = S_3$, (1) may be written

$$\frac{aS_1}{p} + \frac{bS_2}{q} = \frac{cS_3}{r} = 0,$$

and from (3), we at once obtain

$$\frac{\frac{1}{p}}{\begin{vmatrix} bS_2 & cS_3 \\ m & n \end{vmatrix}} = \frac{\frac{1}{q}}{\cdots\cdots} = \frac{\frac{1}{r}}{\cdots\cdots}.$$

Substituting the values of p, q, r in (2), we get

$$\frac{\alpha\left\{\dfrac{1}{b\begin{vmatrix} cS_3 & aS_1 \\ n & l \end{vmatrix}} - \dfrac{1}{c\begin{vmatrix} aS_1 & bS_2 \\ l & m \end{vmatrix}}\right\}}{-\dfrac{1}{\begin{vmatrix} bS_2 & cS_3 \\ m & n \end{vmatrix}} + \dfrac{\cos C}{\begin{vmatrix} cS_3 & aS_1 \\ n & l \end{vmatrix}} + \dfrac{\cos B}{\begin{vmatrix} aS_1 & bS_2 \\ l & m \end{vmatrix}}} + \ldots = 0.$$

on simplifying by the use of the relation $S_1 + S_2 + S_3 = 0$, it is found that both the numerator and denominator contain the common factor $bcl + cam + abn$, and the locus may be written as

$$\frac{\alpha\left(\dfrac{m}{bS_2} - \dfrac{n}{cS_3}\right)}{-\dfrac{a}{S_1} + \dfrac{m \cos c}{S_2} + \dfrac{n \cos B}{S_3}} + \ldots + \ldots = 0. \quad \ldots \quad (4)$$

This equation is of the sixth degree and differs from equation (2) in having $\frac{l}{S_1}, \frac{m}{S_2}, \frac{n}{S_3}$ written respectively for p, q, r.

Questions 661 and 681.

(S. Ramanujan) :—Solve in integers $x^3+y^3+z^3=u^6$
Solve in integers $x^3+y^3+z^3=1$.

Additional Solution and remarks by N. B. Mitra.

To solve in integers $x^3+y^3+z^3=u^6$. ... ... (1)

Let $v=u^2$, then $x^3+y^3+z^3=v^3$. ... ... (2)

In the Journal of the Indian Mathematical Society, Vol. XIII, pp. 15—17, four methods of obtaining solutions of the problem were given.

A method of obtaining the *general* solution is given below :—

Euler gave the general solution of (2) as

$$x=(3ac+3bc-ad+3bd)(d^2+3c^2)-(a^2+3b^2)^2,$$
$$y=(3ac-3bc+ad+3bd)(d^2+3c^2)+(a^2+3b^2)^2,$$
$$z=(d^2+3c^2)^2-(3ac+3bc-ad+3bd)(a^2+3b^2),$$
$$v=(d^2+3c^2)^2+(3ac-3bc+ad+3bd)(a^2+3b^2).$$

Since (2) is homogeneous in x, y, z, v, it is evident that if any values x y, z, v satisfy it, then kx, ky, kz, kv will also satisfy it, where k is any quantity whatever, not necessarily integral.

Put $a^2+3b^2=p$, $d^2+3c^2=q$, $r=3(ac+bd)+(3bc-ad)$,

$$s=-3(ac+bd)+(3bc-da). \quad \ldots \quad (3)$$

Then $kx=qs-p^2$, $ky=p^2-qr$, $kz=q^2-ps$, ... (4)

and $kv=q^2-pr$. ... ... (4′)

Now from (3),

$$pq=(a^2+3b^2)(d^2+3c^2)=(-ad+3bc)^2+3(ac+bd)^2$$
$$=\left(\frac{r+s}{2}\right)^2+3\left(\frac{s-r}{6}\right)^2=\tfrac{1}{3}(r^2+rs+s^2).$$

$\therefore$ $3pq=r^2+rs+s^2$ ... ... (5)

Hence (4) and 4′ constitute the general solution of (2) in terms of 4 parameters p, q, r, s connected by the relation (5) and a fifth one k.

From (4)′, $(q^2-pr=ku^2)$. ... ... (6)

From (5), $(2s+r)^2=3(4pq-r^2)$.

so that $2s+r$ is a multiple of 3; let $2s+r=3t$. ... ... (7)

$\therefore$ $4pq=r^2+3t^2$. ... ... (8)

Hence (4), (6), (7) and (8) constitute the general solution of (1).

Now it is known that factors of numbers of the form r^2+3t^2 can only be of the form $3^m f^2 N$, where m is o or 1, f is any integer and N is entirely composed of factors which are all primes (including 1) of the form $6n+1$.

Hence p and q must both be of this form. ... ... (α)

To solve (8), therefore, give *any* suitable values to p and q consistent with (α) and find r and t. Then (7) gives s. Substituting the values of p, q and r in (6), we get k and u. Then from (4) we get x, y, z. It is to be observed that p and q must be positive but r and s may be either positive or negative. (8) shows that they must be either both odd or both even.

It is understood that if k is not of the form $1/m$, where m is an integer, we may have to replace x, y, z, u by n^2x, n^2y, n^2z, nu (where n has a suitable value) in order to get an integral solution.

Ex. Take $p = 7$, $q = 1$.

Then $r^2 + 3t^2 = 28$, giving $r = \pm 1$, $t = \pm 3$; or $r = \pm 5$, $t = \pm 1$.

Taking $r = 1$, we have from (7), $s = 4$ or -5; and from (6) $ku^2 = -6$, so that we may take $k = -3/2$, $u = 2$.

$\therefore$ $x = 30$ or 36; $y = -32$, $z = 18$ or -24, which give

$$9^3 - 8^3 - 6^3 = 1^6 \quad \ldots \quad \ldots \quad \text{(i)}$$

and

$$30^3 - 32^3 + 18^3 = 2^6 \quad \ldots \quad \ldots \quad \text{(ii)}$$

Taking $r = -1$, we have $s = 5$ or -4 and $ku^2 = 8$, whence $k = 2$, $u = 2$.

Hence $x = -22$ or $-53/2$, $y = 25$, $z = -17$ or $29/2$, giving

$$-22^3 + 25^3 - 17^3 = 2^6 \quad \ldots \quad \ldots \quad \text{(iii)}$$

and

$$-106^3 + 100^3 + 58^3 = 4^6. \quad \ldots \quad \text{(iv)}$$

Taking $r = 5$, we get $s = -2$ or -3 and $ku^2 = -34$, so that we may take $k = -2/17$, $u = 17$, whence $x = 425$, $901/2$; $y = -374$; $z = -68$, $-493/2$.

Hence

$$425^3 - 374^3 - 68^3 = 17^6 \quad \ldots \quad \text{(v)}$$

and

$$1802^3 - 1496^3 - 986^3 = 34^6. \quad \ldots \quad \text{(vi)}$$

Lastly, taking $r = -5$, we have $s = 4$ or 1; $ku^2 = 36$, whence $k = 1$, $u = 6$.

$\therefore$ $x = -45, -48$; $y = 54$; $z = -27, -6$, giving

$$-5^3 + 6^3 - z^3 = 2^6 \quad \ldots \quad \ldots \quad \text{(vii)}$$

and

$$-48^3 + 54^3 - 6^3 = 6^6. \quad \ldots \quad \text{(viii)}$$

Thus this one assumption of the values of p and q gives eight solutions.

Similarly $p = 1$, $q = 3$ will give the following six solutions :—

$$8^3 + 1^3 + 6^3 = 3^6 \quad \ldots \quad \ldots \quad \text{(ix)}$$

$$-10^3 + 1^3 + 18^3 = 3^6 \quad \ldots \quad \ldots \quad \text{(x)}$$

$$-6^3 - 48^3 + 54^3 = 6^6 \quad \ldots \quad \ldots \quad \text{(xi)}$$

$$-15^3 - 12^3 + 18^3 = 3^6 \quad \ldots \quad \ldots \quad \text{(xii)}$$

$$-3^3 + 30^3 + 27^3 = 6^6 \quad \ldots \quad \ldots \quad \text{(xiii)}$$

$$24^3 + 30^3 + 18^3 = 6^6. \quad \ldots \quad \ldots \quad \text{(xiv)}$$

It remains to show that (4), (6), (7) & (8) constitute the *general* solution of (1), (exclusive of trivial solutions). For this purpose it is sufficient to show how to determine p, q, r, s, t and k when x, y, z, u are respectively replaced by n^2x, n^2y, n^2z and nu, where x, y, z and u are any given set of values satisfying (1).

From (4) and (6), we have

$$k(x+y) = q(s-r) \text{ and } k(u^2-z) = p(s-r).$$

$\therefore$ $$p/(u^2-z) = q/x(+y) = \lambda \text{ suppose,}$$

so that $$p = \lambda(u^2-z) \text{ and } q = \lambda(x+y). \quad \dots \quad (9)$$

Also from (4) and (6), $k(qu^2-py) = q^3-p^3$. Substituting from (9) for p and q,

$$\lambda^2 = k\,\frac{u^2(x+y)-p(u^2-z)}{(x+y)^3-(u^2-z)^3}. \quad \dots \quad \dots \quad \dots \quad (10)$$

Here the numerator on the right is homogeneous of degree 2 in x, y, z, u^2 and the denominator of degree 3. Hence (10) will be satisfied by $\lambda=1$, if we replace x, y, z, u^2 by n^2x, n^2y, n^2z, nu and determine n suitably.

Equation (9) will then become $p=(u^2-z)n^2$, $q=(x+y)n^2$. Thus p and q are determined. Similarly, substituting these multiples of x, y, z, in (4), we get linear equations to determine r, s, and k and then (7) affords the value of t.

We have now to show that (8) is satisfied. Substituting in (1), the values of x, y, z, u obtained from (4) and (6), we get

$$\{(qs-p^2)^3+(p^2-qr)^3+(q^2-ps)^3-(q^2-pr)^3\}/k^3=0$$

or $$(p^3-q^3)(r-s)(r^2+rs+s^2-3pq) = 0, \text{ provided } 1/k \neq 0,$$

which obviously is the case except for the solution $x=y=z=u=0$.

If $p=q$, then $x=-z$, $y=u^2$; if $r=s$, then $x=-y$, $z=u^2$.

Hence except in the case of trivial solutions, equation (5) and therefore equation (8) is satisfied.

Thus (4), (6), (7) and (8) afford the general solution of (1).

[*N.B.*—The one parameter solution (ii) given on p. 15, *J. I. M. S.*, Vol. XIII, may obviously be changed into a two parameter one. Thus

$$x = -5k^4 + 20k^3l + 88k^2l^2 - 216kl^3 - 12l^4$$
$$y = 6k^4 - 16k^3l + 8k^2l^2 - 32kl^3 + 184l^4$$
$$z = -3k^4 - 20k^3l + 136k^2l^2 - 40kl^3 - 148l^4$$
$$u = 2k^2 - 4kl + 6l^2.]$$

The above investigation affords a tentative method of solving the equation $x^3+y^3+z^3=1$. (Question No. 681 by Ramanujan).

For, putting $u=1$, we get from (6), $k=q^2-pr$ and from (4)

$$x = \frac{qs-p^2}{q^2-pr},\ y = \frac{p^2-qr}{q^2-pr},\ z = \frac{q^2-ps}{q^2-pr}.$$

If these values are integral, we have a solution of the proposed equation. In particular, if we take $r = 0$, $q = 1$, we get from (8), $4p = 3t^2$ and from (7), $2s = 3t$ and from (6) $k = 1$.

$$\therefore \quad x = \frac{3t}{2} - \frac{9t^4}{16},\ y = \frac{9t^4}{16},\ z = 1 - \frac{9t^3}{8};$$

or writing $2t$ for t, we have $x = 3t(1 - t^3)$, $y = 9t^4$, $z = 1 - 9t^3$ which is the solution given in *J. I. M. S.*, Vol. XIII, p. 15.

Question 993.

(Communicated by Prof. Hembaj):—From a point P perpendiculars PL, PM, PN are drawn to the sides of a triangle ABC. If AL, BM, CN meet at a point Q, find the locus of P and prove that PQ passes through a fixed point.

Solutions by Prof. N. B. Mitra, M.A., and Mr. L. N. Subramaniam, and 'partial' Solution by Mr. Sadanand.

[This question is substantially the same as Question 340 and is solved on page 225, Vol. VI of the Journal.]

Question 1144.

(K. J. Sanjana, M.A.) :—Show that

$$(a)\quad \frac{1}{20}\left\{\frac{20}{3^3 \cdot 4^2} + \frac{47}{6^3 \cdot 5^2} + \frac{84}{10^3 \cdot 6^2} + \frac{131}{15^3 \cdot 7^2} + \dots \infty\right\} = \pi^2 - 9\cdot 8\dot{6}.$$

(The value of R. H. S. is less than ·002938).

$$(b)\quad \frac{1}{25}\left\{\frac{19}{3^2 \cdot 4^2} + \frac{46}{6^2 \cdot 5^2} + \frac{79}{10^2 \cdot 6^2} + \frac{118}{15^2 \cdot 7^2} + \dots \infty\right\} = 9{\cdot}87\dot{8} - \pi^2.$$

(The value of R.H.S. is less than ·009285).

Solution by K. Satyanarayana.

(a) By the method of differences,

$$T_n = \frac{1}{20}\left\{\frac{20+27(n-1)+5(n-1)(n-2)}{\left[3+3(n-1)+\frac{(n-1)(n-2)}{2}\right]^3 (n+3)^2}\right\}$$

$$= \frac{8}{20} \cdot \frac{5n^2 + 12n + 3}{(n+1)^3 (n+2)^3 (n+3)^2}$$

$$= \frac{8}{20} \left\{ \frac{-\frac{21}{2}}{n+1} + \frac{\frac{9}{2}}{(n+1)^2} + \frac{-1}{(n+1)^3} + \frac{6}{(n+2)} + \frac{9}{(n+2)^2} + \frac{1}{(n+2)^3} + \frac{\frac{9}{2}}{(n+3)} + \frac{\frac{3}{2}}{(n+3)^2} \right\}$$

by the ordinary method of partial fractions.

$$\therefore T_n = \frac{8}{20} \left\{ \frac{-\frac{21}{2}}{n+1} + \frac{\frac{12}{2}}{n+2} + \frac{\frac{9}{2}}{n+3} + \frac{-1}{(n+1)^3} + \frac{1}{(n+2)^3} + \frac{\frac{9}{2}}{(n+1)^2} + \frac{\frac{18}{2}}{(n+2)^2} + \frac{\frac{3}{2}}{(n+3)^2} \right\}$$

From which the sum to ∞ is seen to be

$$\frac{8}{20} \left\{ \frac{-\frac{21}{2}}{2} + \frac{\frac{12}{2} - \frac{21}{2}}{3} + \frac{-1}{2^3} + \left(\frac{9}{2} + \frac{18}{2} + \frac{3}{2}\right) \sum \frac{1}{n^2} - \frac{9}{2}\left(\frac{1}{1^2}\right) - \frac{18}{2}\left(\frac{1}{1^2} + \frac{1}{2^2}\right) - \frac{3}{2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\right) \right\}$$

$$= 6 \sum \frac{1}{n^2} - 9{\cdot}8\dot{6} = \pi^2 - 9{\cdot}8\dot{6}.$$

(*b*) Similarly $T_n = \frac{1}{25}\left\{ 4 . \frac{19 + 27(n-1) + 3(n-1)(n-2)}{(n+1)^2 (n+2)^2 (n+3)^2} \right\}$

$$= \frac{4}{25} \cdot \frac{3n^2 + 18n - 2}{(n+1)^2 (n+2)^2 (n+3)^2}$$

$$= \frac{4}{25} \left\{ \frac{\frac{63}{4}}{n+1} + \frac{-\frac{17}{4}}{(n+1)^2} + \frac{\frac{24}{4}}{n+2} + \frac{-26}{(n+2)^2} + \frac{-\frac{87}{4}}{n+3} + \frac{-\frac{29}{4}}{(n+3)^2} \right\}$$

$$= \frac{16}{100} \left\{ \frac{\frac{63}{4}}{n+1} + \frac{\frac{24}{4}}{n+2} + \frac{-\frac{87}{4}}{n+3} + \frac{-\frac{17}{4}}{(n+1)^2} + \frac{-\frac{104}{4}}{(n+2)^2} + \frac{-\frac{29}{4}}{(n+3)^2} \right\}$$

Hence sum

$$= \frac{16}{100} \left\{ \frac{\frac{63}{4}}{2} + \frac{\frac{24}{4} + \frac{63}{4}}{3} + \left[-\frac{17 + 104 + 29}{4} \right] \sum \frac{1}{n^2} + \frac{17}{4}\left(\frac{1}{1^2}\right) . + \frac{104}{4}\left(1 + \frac{1}{2^2}\right) + \frac{29}{4}\left(1 + \frac{1}{2^2} + \frac{1}{3^2}\right) \right\}$$

$$= 9{\cdot}87\dot{8} - 6 \sum \frac{1}{n^2} = 9{\cdot}87\dot{8} - \pi^2.$$

Taking $\pi^2 = 9{\cdot}8696044008\ldots$,

R.H.S. in (*a*) $= {\cdot}0029378008\ldots$ *i. e.* $< {\cdot}002938$,

R. H. S. in (*b*) $= {\cdot}0092844880\ldots$ *i. e.* $< {\cdot}009285$.

QUESTIONS FOR SOLUTION.

1219. (F. H. V. GULASEKHARAM) :—If d is the length of the diameter of the pedal circle of a pair of isogonal points P, P′ *w. r. t.* a triangle ABC, and t, t' the lengths of the tangents from P, P′ to the polar circle of ABC, prove

$$d^2 = t^2 + t'^2.$$

1220. (R. VAIDYANATHASWAMI) :—Shew that the necessary and sufficient condition that each of two given quadrics P, Q reciprocates the other into itself, is that the determinant | P + λ Q | has only linear invariants-factors and only two distinct characteristic roots, where P, Q denote the determinants of the coefficients of the two quadrics.

As a particular case, shew that two quadrics in three dimensions, which are each its own reciprocal *w.r.t.* the other, must either have ring-contact or four common generators.

1221. (R. VAIDYANATHASWAMI) :—If $2^n = \mu$, shew that in space of n dimensions, there exist, associated with each quadric Q_1, sets of $\mu - 1$ other quadrics $Q_2\ Q_3 \ldots Q_\mu$ having the following properties ;

(1) Any two quadrics of a set $Q_1\ Q_2 \ldots Q_\mu$ are, each its own reciprocal *w.r.t.* the other.

(2) Any two quadrics of a set touch at all their common points, and their locus of contact splits into two distinct quadric loci lying respectively in two flat sub-regions of r and $(n - r - 1)$ dimensions. This includes $r = 0$ as a possible case; when $r = 0$ the two quadrics touch along a single quadric lying in a sub-region of $n - 1$ dimensions.

(3) All the quadrics $Q_1\ Q_2 \ldots\ldots Q_\mu$ have a common self-polar set of $n + 1$ points; and any other quadric Q having the same self-polar set is reciprocated by each of the quadrics Q_r into the same quadric Q'.

(4) The pencil $(\lambda Q_r + Q_s$ possesses only two distinct cones; if these are $\lambda_1 Q_r + Q_s$ and $\lambda_2 Q_r + Q_s$, then each of the quadrics Q_r, Q_s reciprocate $\lambda Q_r + Q_s$ into $\lambda' Q_r + Q$, where $(\lambda\lambda', \lambda_1 \lambda_2)$ is a harmonic range. Further $\lambda_1 + \lambda_2$ must be equal to zero.

1222. (A. KRISHNASWAMI AIYANGAR) :—If $a_1 (1 + a_2) = a_2 (1 + a_3) = a_{n-1} (1 + a_n) = a_n (1 + a_1)$, prove that each is equal to $-\frac{1}{4} \sec^2 \frac{r\pi}{n}$ where r may take any value from 1 to $\frac{n-2}{2}$ if n is even, or any value from 1 to $\frac{n-1}{2}$ when n is odd.

1223. (A. A. KRISHNASWAMI AIYANGAR) :—Prove that the orthoptic locus of the first negative pedal of an ellipse with respect to any point on its circumference is a similar ellipse.

THE INDIAN MATHEMATICAL SOCIETY.

Summary of Accounts for the year 1921.

Receipts :—	Rs.	A.	P.	**Expenditure** :—	Rs.	A.	P.
Opening balance ...	5403	13	6	Ordinary working expenses ...	292	14	0
Subscription from members ...	3373	8	9	Books and Journals ...	141	11	0
Subscriptions for Journal ...	457	0	0	Printing Journal ...	958	0	6
Interest on Investments ...	298	13	8	Library expenses ...	200	0	0
Miscellaneous ...	37	10	0	Closing balance ...	*7978	4	5
	9570	13	11		9570	13	11

	Rs.	A.	P.
* Madras Central Urban Bank	3500	0	0
Indian Bank ...	3069	13	3
Bombay Savings Bank ...	400	0	0
Treasurer ...	822	2	11
Secretary ...	64	2	0
Assistant Secretary ...	108	0	0
Librarian ...	0	4	0
Joint Editor ...	13	14	3
	7978	4	5

MADRAS,
13th April 1922.

S. NARAYANA AIYAR.
Hon. Treasurer.